ON THE SPERNER LEMMA AND ITS APPLICATIONS

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Abstract. This paper presents elementary combinatorial proofs of Sperner's Lemma and discusses several non-trivial theorems to which it has been applied. Examples are the Brouwer Fixed Point Theorem, the Fundamental Theorem of Algebra, and the solution to the Cake-cutting problem.

1. Introduction

At first blush, the Sperner Lemma seems to be an almost obvious fact about labelings of vertices on a special type of graph. Indeed, its proof is a fairly simple one, first established by the German mathematician, Emanuel Sperner in the mid 1900's. However, it has many surprising applications. The most well-known application is an elegant proof of the Brouwer Fixed Point theorem without using advanced topics like degree theory or homology.

Before beginning, it is necessary to first introduce some terminology. I will then prove the lemma, then discuss a generalization of it, and several interesting applications.

2. Terminology

An n-simplex is an n-dimensional generalization of a triangle. So a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex, a triangle, a 3-simplex, a tetrahedron, and so forth.

Definition 2.1. In general, if there are $n+1$ independent points $p_0, p_1, \ldots, p_n$ in some Euclidean space, $\mathbb{R}^m$ $(m \geq n)$, then an n-simplex, Δ is the convex hull of this set. Δ can also be expressed as the set $\{p_0, p_1, \ldots p_n\}$. An n-simplex which is written as an ordered $(n+1)$-tuple, $< p_0, p_1, \ldots p_n >$, is called an oriented n-simplex (Unless otherwise mentioned, an n-simplex should be considered oriented for the rest of the paper).

In particular, every point x of the simplex Δ can be expressed as a linear combination of these points:

$$x = \sum_{i=0}^{n} \alpha_i p_i$$

where $\sum_{i=0}^{n} \alpha_i \leq 1$ and $\alpha_i \geq 0$ for each i. We call the coefficients of this linear combination the *barycentric coordinates* for the point x. Every n-simplex Δ is uniquely determined by its $n+1$ vertices. Note that every $k+1$ subset of these $n+1$ vertices also determines a unique k-simplex, $\Delta' \subset \Delta$. Each such 'sub-simplex' is called a *k-face* of Δ.

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Definition 2.2. Suppose there is a finite collection of n-simplices whose union is all of Δ, with the property that if any two of these simplices intersect, they must do so at an entire face common to both simplices. Then we say that the collection gives a *triangulation* of Δ. Any simplex in this collection will be referred to as an *elementary n-simplex*, or simply as an n-simplex within the triangulation.

The concept of triangulation is central to the statement of the Sperner Lemma and is also a very useful tool in the various topological results that will be discussed, because it gives us a way to break a space into very small parts and to use continuity arguments.

We would also like to have terminology concerning the boundary of an oriented simplex. Call any $(n-1)$-face of an n-simplex Δ, a *facet*. For example, if Δ is a tetrahedron, then the facets of Δ are simply the four triangles which form the set boundary of Δ. The following definitions are motivated by some basic concepts in homology, where one forms formal linear combinations of simplices.

Definition 2.3. Given a triangulation of Δ, we say that the oriented elementary simplices, Δ_1 and Δ_2 are equal if their vertices are the same up to an even permutation. If they are the same up to an odd permutation, then we say that $\Delta_1 = -\Delta_2$.

Definition 2.4. If there is a triangulation of a simplex Δ, then an integer linear combination of k-simplices in the triangulation is called a *k-chain.* More generally, a k-chain is any integer linear combination of k-simplices.

The boundary $\partial(S)$ of a k-simplex $S =< p_0, p_1, \ldots, p_k >$ is defined by the following formula (see [8] for a full treatment):

$$\partial(S) = \sum_j (-1)^j < p_0, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k >$$

Notice that this definition of a boundary map fits our intuition of what it should be, for the boundary of 1-simplex (or an edge) is simply its two endpoints, and the boundary of a 2-simplex (triangle) is a linear combination of its three edges. That is,

$$\partial(< p_0, p_1 >) =< p_1 > - < p_0 >$$

and,

$$\begin{aligned}\partial(< p_0, p_1, p_2 >) &= < p_1, p_2 > - < p_0, p_2 > + < p_0, p_1 > \\ &= < p_1, p_2 > + < p_2, p_0 > + < p_0, p_1 >\end{aligned}$$

We can take boundaries of chains by extending this definition linearly. Notice that if a simplex is triangulated, the boundary of the chain of all elementary simplices of the triangulation is simply the boundary of the simplex since all the interior boundaries cancel in the definition. It is straight-forward to check that a boundary of a k-chain has no boundary itself (i.e., $\partial\partial = 0$).

Now consider an n-simplex S with a triangulation in which each vertex is labeled by elements from $\{0, 1, \ldots, n\}$. Denote the label of a vertex q by $L(q)$. We say that the labeling L is *Sperner* (or that S is equipped with a *Sperner labeling* L) if the following two conditions hold:

(1) Each corner of the simplex S is labeled distinctly.
(2) The label of any vertex within the triangulation that lies on a facet of S matches one of the labels of the corners of that same facet.

We say that a labeling of an elementary simplex in the triangulation is *complete* if each of its vertices takes on a distinct label.

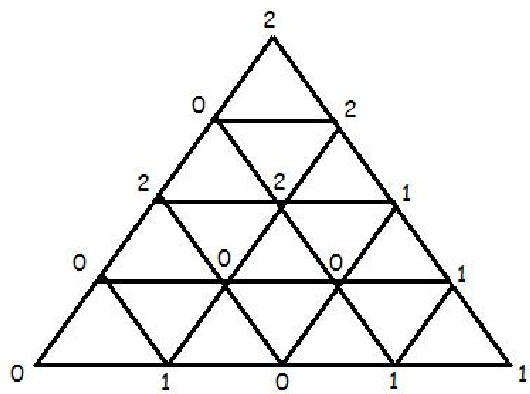

FIGURE 1. Sperner-labeled Triangulation

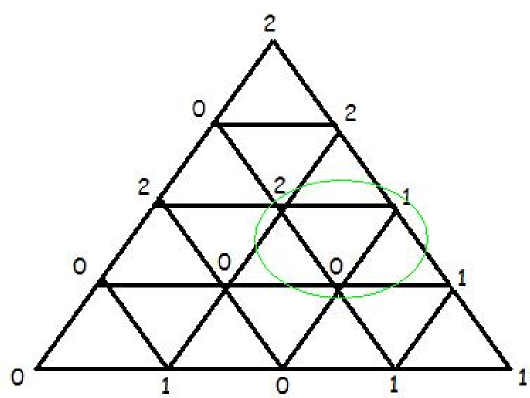

FIGURE 2. A Completely Labeled Simplex

3. SPERNER'S LEMMA

Theorem 3.1 (Sperner's Lemma)**.** *Given any triangulation of an n-simplex which is Sperner-labeled, there exists an odd number of completely labeled elementary n-simplices. In particular, there exists at least one such elementary simplex within the triangulation.*

Before giving the general case proof, I will discuss some low dimensional cases. The one-dimensional case is used to prove the two-dimensional case, which will be generalized in an inductive way to hold for n-dimensional simplices.

3.1. **The One-Dimensional Case.** In this trivial case of the Sperner Lemma, there is an interval of the real line which has been subdivided into $k-1$ subintervals . The k vertices are labelled with either 0 or 1, with the stipulation that the boundary points are labelled differently. Without loss of generality, we can assume that the interval is [0,k-1] with the vertices placed at integer points. I will denote the subintervals of the form $[i, i+1] \subset [0, k-1]$ by the symbol σ_i. They are the 1-simplices in the triangulation of $[0, k-1]$. Let the (a, b)-simplices denote the σ_i which are labelled a on one vertex and b on the other, with no restriction on ordering. Sperner's Lemma states that the number of (0,1)-simplices are odd.

FIGURE 3. Sperner's Lemma in one dimension

Proof. For each σ_i, let $F(\sigma_i)$ be the number of endpoints of σ_i which are labelled by zero. F takes the values 0,1 and 2. The proof proceeds by calculating the value of $\sum_i F(\sigma_i)$ in two ways. First we note that this value is equal to the number of (0,1)-simplices, plus twice the number of (0,0)-simplices. In particular, it is the number of (0,1)-simplices plus an even number. We can compute the sum in a different way by noticing that every vertex in the interior of $[0, k-1]$ which is labelled zero contributes 2 to the sum. Therefore, the sum is also twice the number of vertices labelled zero inside the interval, plus one for the one boundary vertex which is labelled by zero. Since this second method of counting shows that the sum is odd, the number of (0,1)-simplices must also be odd. □

3.2. **The Two-Dimensional Case.** We now consider a simplicial subdivision (triangulation) of a 2-simplex, K, which is just a triangle (see figures 1,2). If the vertices in the triangulation are Sperner labelled by 0,1, and 2, Sperner's Lemma states that there exists an odd number of (0,1,2)-simplices.

Proof. The strategy of the proof is exactly the same as before. Denote the 2-simplices in the triangulation by σ_i. We define a function F on the σ_i by setting $F(\sigma_i)$ to be the number of (0,1)-simplices of the three edges of σ_i. The first way to compute $\sum_i F(\sigma_i)$ goes as follows:

$$\begin{aligned}\sum_i F(\sigma_i) &= 1 \times (\#\{\text{(0,1,2)-simplices}\}) \\ &\quad +2 \times (\#\{\text{(0,1,0)-simplices}\}) \\ &\quad +2 \times (\#\{\text{(0,1,1)-simplices}\})\end{aligned}$$

Notice that this shows that if $\sum_i F(\sigma_i)$ is odd, then the number of (0,1,2)-simplices must also be odd. The second way to calculate this quantity will show that it is indeed odd. The method is again to use the fact that any (0,1)-simplex on the interior of K is a face of exactly two 2-simplices of the triangulation of K. We have that:

$$\begin{aligned}\sum_i F(\sigma_i) &= 2 \times (\#\{\text{(0,1)-simplices in } int(K)\}) \\ &\quad +1 \times (\#\{\text{(0,1)-simplices on } \partial K\})\end{aligned}$$

By the conditions of a Sperner-labelling, only one edge of K can contain (0,1)-simplices. This edge meets the conditions of the one-dimensional case of Sperner's Lemma, and so applying the previous result, we have that the number of (0,1)-simplices on the boundary of K is odd. This shows that in fact, the number of (0,1,2)-simplices in the triangulation must be odd. □

3.3. **The General Case.** The proof of the two-dimensional case generalizes in a natural way to n dimensions. We have already established that Sperner's Lemma is true for one dimensional simplices. Suppose that it holds for $n-1$ dimensions; we would like to see that this implies the lemma for n-simplices. Sperner labelings are formulated in a way which is very conducive to using induction because each facet of a triangulated, Sperner-labeled n-simplex is an $n-1$-simplex which inherits triangulation and a Sperner-labeling.

Let K be a triangulated n-simplex with a Sperner labeling. Denote the elementary n-simplices in the triangulation by σ_i. Define F on the σ_i by setting $F(\sigma_i)$ to be the number of $(0, 1, \ldots, n-1)$-simplices of the $n+1$ facets of σ_i. As before, we compute $\sum_i F(\sigma_i)$.

First,

$$\begin{aligned}\sum_i F(\sigma_i) &= 1 \times (\#\{\text{(0,1,\ldots,n)-simplices}\}) \\ &\quad +2 \times (\#\{\text{(0,1,\ldots,n-1,0)-simplices}\}) \\ &\quad +2 \times (\#\{\text{(0,1,\ldots,n-1,1)-simplices}\}) \\ &\quad \vdots \\ &\quad +2 \times (\#\{\text{(0,1,\ldots,n-1,n-1)-simplices}\})\end{aligned}$$

The second way of computing $\sum_i F(\sigma_i)$ will show that it is odd. Using the fact that every $(0, 1, \ldots, n-1)$-simplex on the interior of K is a face of precisely 2 n-simplices in the triangulation of K (this can be shown by simple geometric arguments):

$$\begin{aligned}\sum_i F(\sigma_i) &= 2 \times (\#\{\text{(0,1,\ldots,n-1)-simplices in } int(K)\}) \\ &\quad +1 \times (\#\{\text{(0,1,\ldots,n-1)-simplices on } \partial K\})\end{aligned}$$

Now only one facet of ∂K contains $(0, 1, \ldots, n-1)$-simplices, and that number is odd by induction, and so the number of $(0, 1, \ldots, n)$-simplices is odd.

3.4. **A Generalized Version of Sperner's Lemma.** I will now discuss a generalization of the lemma, which can be used to prove the Fundamental Theorem of Algebra. None of the other applications rely on the generalization however.

Sperner's Lemma can be generalized for any space which admits a triangulation (not just on a triangulated simplex) with no restrictions on how the vertices are labeled. For an introductory treatment of these spaces (simplicial complexes), see Armstrong ([2]). The generalization provides an interesting perspective because it connects the lemma with the topology and orientation of the space.

Definition 3.2. Let $q_0, q_2, \dots, q_k$ be a given ordered list of non-negative integers. If the $q_0, \dots, q_k$ happen to be a permutation of $0, 1, \dots, k$, then we define $N(q_0, q_1, \dots, q_k)$ to be $+1$ if the permutation is even, and -1 if the permutation is odd. If it is not a permutation, then define $N(q_0, q_1, \dots, q_k)$ to be zero.

For example,

$$N(0,1,2,3) = N(1,2,0,3) = N(1,0,3,2) = 1$$

$$N(0,1,3,2) = N(3,0,1,2) = N(0,2,1,3) = -1$$

$$N(0,0,1) = N(2), N(1,2179) = 0$$

Now suppose $\Delta =< p_0, p_1, \dots, p_k >$ is an oriented k-simplex, and each vertex is assigned a label, $L(k)$. We can define N for oriented simplices as:

$$N(\Delta) = N(L(p_0), L(p_1), \dots, L(p_k))$$

N can also be extended to n-chains linearly, so if $C = \sum_j \alpha_j \Delta_j$, then $N(C) = \sum_j \alpha_j N(\Delta_j)$.

Theorem 3.3 (Generalized Sperner)**.** *Suppose C is a labeled k-chain of k-simplices with labels chosen from $\{0, 1, \dots, k\}$. Then,*

$$N(C) = (-1)^k N(\partial(C))$$

Remark 3.4. Why is this a generalization of Sperner's Lemma? In the case that C is a triangulated n-simplex equipped with a Sperner-labeling, let F^k denote the k-face of C (with inherited triangulation and labeling) with corner vertices labeled with $0, \dots, k$. By the requirements of the Sperner-labeling, $C = F^n$. Applying the generalized lemma repeatedly, we have:

$$N(C) = N(F^n) = \pm N(F^{n-1)}) = \pm N(F^{n-2)}) = \cdots = \pm N(F^{0)}) = \pm 1$$

And this is enough to show the existence of a completely labeled elementary n-simplex in the triangulation.

Proof. This proof is based on the proof found in [3]. By linearity, it suffices to show that the theorem holds for just one k-simplex $\Delta =< p_0, p_1, \dots, p_k >$. That is, we would like to see that the following is true:

$$N(q_0, q_1, \dots, q_k) = (-1)^k \sum_{j=1}^{k} (-1)^j N(q_0, \dots, q_{j-1}, q_{j+1}, \dots, q_k)$$

where $(q_0, q_1, \dots, q_k)$ is the labeling on the vertices. We can associate a 'permutation' matrix with the labeling $(q_0, q_1, \dots, q_k)$. Let P be this $(k+1) \times (k+1)$ matrix defined by letting $P_{ij} = 1$ if $q_i = j$, and $P_{ij} = 0$ if not. Since N is defined to be equal to the sign of a permutation, it immediate that $\det(P) = N(q_0, q_1, \dots, q_k)$. Of course, P is not *really* a permutation if $(q_0, q_1, \dots, q_k)$ is not a permutation, but the equality between the determinant and N still holds.

Now we construct a new matrix P' which is equal to P except that we replace the last column by all ones. Note that $(-1)^k \sum_{j=1}^{k} (-1)^j N(q_0, \ldots, q_{j-1}, q_{j+1}, \ldots, q_k)$ is simply the determinant of P' by expanding by the last column.

It now suffices to show that $\det(P) = \det(P')$. But this is true since we can reduce P to P' by adding all of the columns of P to its last column.

□

4. Applications

Even though Sperner's lemma is proved in a discrete setting, it is often applied in continuous situations. The general strategy used in the following sections is:

(1) Triangulate a compact and convex region $K \subset \mathbb{R}^n$.
(2) Label the vertices of the triangulation in some clever way.
(3) Apply Sperner's Lemma to find a completely labeled simplex.
(4) Take finer and finer triangulations of the space, and use continuity arguments to deduce some result in the limiting case.

The three main applications that will be discussed are the Brouwer Fixed-Point theorem, the Cake-Cutting/Rental Harmony problem, and the Fundamental Theorem of Algebra.

4.1. **The Brouwer Fixed Point Theorem.** The celebrated Brouwer Fixed Point Theorem is a surprising result which was first proved by L. Brouwer in the early 1900s. Informally, the theorem is sometimes explained using a cup of coffee which is being stirred (in a continuous fashion). The theorem states that after stirring, there is at least one point of the coffee which has returned to its original position before stirring. More formally,

Theorem 4.1 (Brouwer Fixed-Point Theorem)**.** *Let K be a compact and convex set in $\mathbb{R}^n$. Suppose $f : K \to K$ is a continuous map. Then there exists some $x \in K$ such that $f(x) = x$.*

In fact, using degree theory arguments, the theorem can be generalized to sets K which are homeomorphic to a disk, but I will not discuss this generalization here. The general procedure for the proof of the Brouwer Fixed-Point theorem is to first show that it is true for n-simplices (with the aid of the Sperner Lemma). Then it is a simple task to extend the result to the general case.

The proof of Brouwer's Fixed Point Theorem is nonconstructive [1]. Suppose for the sake of contradiction that there exists no fixed point of of f. We can define a continuous retraction $r(x)$ from K to its boundary ∂K by the following. We take a ray originating $f(x)$ in the direction pointing to x. Let T_x be the smallest $t \geq 1$ such that $f(x)+t(x-f(x)) \in \partial K$. Then define $r(x) = f(x)+T_x(x-f(x))$. Observe that $r(x)$ is well-defined since $x \neq f(x)$ by assumption. It is also clear that $r(x)$ fixes any point on the boundary, and that it is continuous, by the continuity of f. Hence it is a continuous retraction of K to its boundary. The following theorem will show that such an $r(x)$ cannot possibly exist.

Theorem 4.2 (Retraction Mapping Theorem)**.** *Let $K \subset \mathbb{R}^n$ be compact and convex (and nonempty). There is no continuous retraction from K to its boundary ∂K.*

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[1] Interestingly, in his later years, Brouwer became a promoter of the Intuitionist movement, which did not accept nonconstructive proofs. Brouwer actually renounced his own theorem.

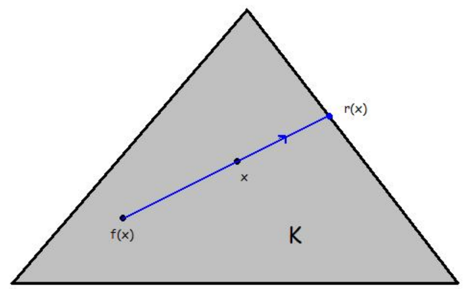

FIGURE 4. Retraction from K to ∂K

Proof. (based on [6]) For simplicity of exposition, I will prove it for $n = 2$, but the proof generalizes in a straight forward manner to arbitrary dimensions. The procedure is to first show that this is true if K is an equilateral triangle (say, with side lengths equal to 1) and then use the result to jump to the general case.

Consider a continuous map $g : K \to \partial K$ which is the identity when restricted to the boundary. Label the vertices of K by the numbers $0, 1$ and 2. For any $x \in K$, we may associate a labeling of x by setting $S(x)$ to the label of the vertex which is closest to $g(x)$ (ties can be broken arbitrarily). Notice that on the boundary of K, any point is labeled with the label of one of the vertices which determine the edge on which it lies, since K is equilateral.

Since g is continuous, there exists some $\delta > 0$ such that $x, y \in K$ satisfying $|x - y| < \delta$ implies that $|g(x) - g(y)| < \frac{\sqrt{3}}{4}$. We now triangulate K so that for any triangle Δ in the subdivision, all edge lengths of Δ are less than $\frac{\sqrt{3}}{4}$. This triangulation, inherits a Sperner labeling from $S(x)$, and so we can apply Sperner's lemma, which guarantees the existence of at least one small triangle within the triangulation whose vertices are labeled 0,1,2. Since all of the vertices in this triangle are within δ of each other, for any pair of these vertices, $\{x, y\}$, we have that $|g(x) - f(y)| < \frac{\sqrt{3}}{4}$. This inequality together with the fact that each vertex is labeled differently implies that the retraction g maps every pair of these vertices to points on one edge of K, and every pair of points gets mapped to a different side. Since this is impossible, we have the desired contradiction which shows that g cannot be continuous.

Now suppose in the general case that K is compact and convex, and $g : K \to \partial K$ is a continuous retraction. By the Heine-Borel theorem, K is bounded, and so one can find a large enough equilateral triangle Γ which contains K. Choose a point $p \in int(K)$. For points in $\Gamma \setminus int(K)$, one can use p to define a map F to the boundary $\partial\Gamma$ by the following. If $x \in \Gamma \setminus int(K)$, set $F(x)$ to be the point on $\partial\Gamma$ where the ray originating at p going in the direction of x intersects $\partial\Gamma$. Note that F is well-defined since $p \notin \Gamma \setminus int(K)$, and F is continuous. Now we can extend F to be a function ϕ over all of Γ by mapping a point $x \in K$ to $F(g(x))$. This extension of course remains continuous by continuity of g. ϕ is therefore a continuous retraction from Γ to $\partial\Gamma$ constructed from g. Applying the above result shows that ϕ cannot exist. Consequently, g cannot exist. $\square$

An interesting point of note is that it was shown in 1974 by Yoseloff ([5]) that Brouwer's Fixed-Point Theorem actually implies Sperner's Lemma, so the statements are in fact equivalent. In [4], Jarvis and Tanton give a proof of the well-known Hairy Ball theorem using the Sperner Lemma, which is often proved via the Brouwer Fixed-Point Theorem. This theorem states (informally) that it is impossible to comb the hairs on a fuzzy ball in a continuous way such that each hair is tangent to the surface of the ball.

4.2. **Cake Cutting.** The Cake Cutting problem is an example of a fair-division, or dispute resolution question. These types of problems often deal with the problem of dividing some quantity among several people, so that everyone remains happy with the outcome. Similar problems include chore-division, where one seeks to divide a list of chores in a fair way, and rental harmony, where n people rent a house with n rooms, and one seeks to somehow partition the rent in some way such that each person prefers a different room. In these problems, one would like to know when a solution exists, and if a solution does indeed exist, how can it be found?

The Cake-Cutting problem, in particular, was introduced in 1947 by Steinhaus. Suppose that there is a rectangular cake that must be distributed among n people. To cut the cake, we can use $n-1$ knives to make $n-1$ cuts parallel to the left edge of the cake. Everyone may have differing opinions on what makes a slice valuable. For example, one person might prefer a slice with more yellow frosting, or one person might just prefer a really large slice. Yet another person may prefer a slice with fruit. The goal is to cut in a way so that each person is appeased. In the end, we would like a person to be able to measure the worth of a slice of cake by their own scale, independent to how the others measure theirs. In order for solutions to exist however, we must set some assumptions on how people compare the cake slices.

Definition 4.3. Suppose a cake is partitioned into slices by some cut-set. We say that a person *prefers* some given slice, if he thinks that no other slice in the cut-set is better. Notice that anybody's preferred slice is independent of the preferences of other people. Furthermore, this definition guarantees that everyone always prefers at least one slice given any cut-set.

We assume the following:

(1) People will always choose any slice over nothing at all. That is, each of the slices is in some way minimally acceptable to everyone, and everyone is hungry.
(2) We also make the reasonable assumption that if a person prefers some slice belonging to a convergent sequence of cut-sets, then that person prefers the same slice in the limiting cut-set. Note that convergence is a notion that makes sense as any sequence of cut-sets can be thought of as a sequence of points in $\mathbb{R}^n$.

Observe that any way of cutting a cake is fully determined by the relative lengths of the pieces. Therefore, we can represent such a 'cut-set' by an n-tuple of non-negative real numbers. Now, without loss of generality, we can assume that the length of the cake is 1, and so a cut-set is expressible as $(x_1, x_2, \ldots, x_n) \in [0,1]^n$ with $\sum_{i=1}^{n} x_i = 1$, where the size of the kth slice is given by x_k. The space of all possible cut-sets therefore forms an $(n-1)$-simplex in $\mathbb{R}^n$. We are now ready to prove the theorem.

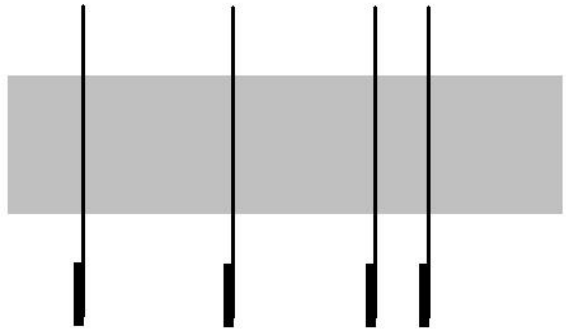

FIGURE 5. Cut-Set of a Cake

Theorem 4.4 (Simmons)**.** *With the above assumptions, there exists a way to divide the cake such that each person prefers a different piece .*

Proof. I will prove the theorem in the case that the cake must divided amongst 3 people named Abigail, Bob, and Chris. The task is to partition a cake of length 1, with 2 cuts. As above, the space S of all possible cut sets is given by:

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1 \text{ and all } x_i \geq 0\}$$

Geometrically, S is the triangle formed by the intersection of the plane $x+y+z=1$ and the first octant of $\mathbb{R}^3$.

We now give S a triangulation and give each vertex a preliminary labeling by the letters A, B, C, corresponding to Abigail, Bob and Chris respectively. The requirement on this preliminary labeling is that each triangle in the subdivision must be an (A, B, C)-simplex. It is easy to see that such a labeling can be realized.

Given this labeling and triangulation, we can now construct another labeling on the triangulation by the numbers 1,2 and 3. We say that the owner of a vertex is the person who corresponds to the label of the vertex given by the preliminary labeling. For example, if some vertex is labeled B, then Bob is the owner of that vertex. The new labeling is constructed in the following way. Let v be any vertex in the triangulation. v describes some cut-set, so we ask its owner which slice (1,2 or 3) in this particular cut-set he or she prefers, and relabel v with this number.

The claim is that this new labeling of S is actually a Sperner labeling. First, at the vertex $(1, 0, 0) \in S$, we see that slice 1 contains the entire cake, and slices 2 and 3 are empty. By our first assumption, this implies that the owner of $(1, 0, 0)$ will definitely choose slice 1 over the others. Therefore, $(1, 0, 0)$ is labeled by 1. Similarly, $(0, 1, 0)$ is labeled by 2, and $(0, 0, 1)$ is labeled by 3. Next, observe that for vertices on each edge of S, there is always a coordinate equal to zero, which means that one of the slices is empty, so that the owners of these vertices always choose the other slices. Thus, each side of S is only labeled by the labels assumed by its endpoints. And this shows that the labeling satisfies the Sperner condition.

Applying Sperner's Lemma, we obtain the existence of a $(1, 2, 3)$-simplex Δ in the triangulation of S. Recall that by the preliminary labeling on S, Δ is also an (A, B, C)-simplex. If Δ is small, then this means that we have found 3 cut-sets that are very similar, where Abigail, Bob and Chris chose different slices.

To get just one cut-set where they choose different slices within the cut-set, we consider this procedure for a sequence of finer and finer triangulations of S. Since each triangle is compact and their sizes decrease according to the sequence, one can find a convergent subsequence of triangles, which, in the limit, become just one point x. By our second assumption, x is a cut-set for which Abigail, Bob and Chris each prefer a different slice of the cake.

□

One point which I have not adequately explained in this proof is how to obtain a sequence of triangulations which get progressively finer. For a two-dimensional S (the case of three people), it is fairly easy to see how to obtain one. For n-dimensions, one method that works is called *barycentric subdivision.* See [2] for an introduction to barycentric subdivision. Essentially, this method takes a simplicial complex K and subdivides by adding in the barycenters of all the simplices in K (in all dimensions) as new vertices, and adding new simplices as necessary to form a new simplicial complex K^1. If $\mu(K)$ is the maximum over the diameters of all the simplices in K, and the $dim(K) = n$, then it can be shown that

$$\mu(K^1) \leq \frac{n}{n+1}\mu(K)$$

This shows that with successive barycentric subdivisions, the diameters of the simplices do indeed approach zero. Using barycentric subdivisions, the above solution to the Cake-Cutting dilemma can be easily extended to dividing a cake for an arbitrary number of people.

In [3], Su gives a solution to a similar problem, called the Rental Harmony problem, which is more difficult. In it, one is presented with the challenge of partitioning house rent among n people who wish to rent a house with n rooms in a fair way. As in the cake-cutting problem, each person may have their own preferences - one might prefer a good view, while another might want a closet for example. The assumptions made about the problem are similar to the ones made in Cake-Cutting, but the problem has a new twist. The main difference is that the rooms are indivisible, and the rents are attached to specific rooms. Su's elegant solution involves applying Sperner's lemma not on a triangulation of the space of all possible rental payment assignments, but its dual.

4.3. **The Fundamental Theorem of Algebra.** The Fundamental Theorem of Algebra is an important result about polynomials. It says that the field $\mathbb{C}$ of complex numbers is algebraically closed. In other words, it guarantees the existence of roots for every non-constant complex polynomial and allows us to express each one as a product of linear terms. Since Gauss submitted the first proof in 1799 as his doctoral dissertation, it has opened itself to a variety of different attacks.

While all of its proofs necessarily contain elements of analysis, many of the proofs carry the influence of another area of mathematics. For example, there is a proof which one studies in Galois Theory. Or, one might consult the proof from algebraic topology which uses results from the theory of fundamental groups. Sperner's lemma provides a nice combinatorial solution based on a discretization of the winding number proof from complex analysis. For the proof, standard facts from real and complex analysis will be assumed to be true (see Ahlfors [7]).

I will first establish some preliminaries, then give a proof of the theorem based on [3]. We first partition the complex plane into three tridrants. For $j = 0, 1, 2$, let:

$$R_j = \{z \in \mathbb{C} : \frac{2\pi}{3} j \leq \arg(z) < \frac{2\pi}{3}(j+1)\}$$

Lemma 4.5. *Suppose $z_j \in R_j$ for $j = 0, 1, 2$, and suppose that for some given ϵ, $|z_j - z_k| < \epsilon$ holds for $j, k \in \{0, 1, 2\}$. Then for each j, $|z_j| < \frac{2}{\sqrt{3}}\epsilon$*

Proof. Without loss of generality, consider z_0. Since z_1, z_2 are in different tridrants, at least one of them, say z_1, satisfies:

$$|\arg(z_0) - arg(z_1)| \geq \frac{\pi}{3}$$

The point on the subspace spanned by z_1 which minimizes the distance to z_0 is the projection of z_1 onto this subspace. This distance $\leq |z_0| \sin(\frac{\pi}{3})$. Thus,

$$|z_0| \sin(\frac{\pi}{3}) \leq |z_0| \sin(|\arg(z_0) - \arg(z_1)|) \leq |z_0 - z_1| < \epsilon$$

Rearranging, we have:

$$|z_0| \leq \frac{\epsilon}{\sin(\frac{\pi}{3})} = \frac{2}{\sqrt{3}}\epsilon$$

□

Theorem 4.6 (Fundamental Theorem of Algebra)**.** *Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a monic polynomial with degree $n \geq 1$. Then there exists some $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. Define a function (labeling) over the complex plane, $\phi : \mathbb{C} \to \{0, 1, 2\}$ where $\phi(z) = j$ if $p(z) \in R_j$. Now for a large enough disk D about the origin, we have that $p(z) \sim z^n$ on the boundary of K (specifically, the winding numbers of $p(z)$ and z^n as z moves counterclockwise along the boundary of the disk are the same).

By viewing $\mathbb{C}$ as $\mathbb{R}^2$, we can form a 'triangulation' of D by inscribing a polygon inside its boundary and forming a triangulation of the polygon. We can get closer and closer to filling out the entire disk by forming polygons with more edges and with finer triangulations. Of course any triangulation of D inherits a labeling from ϕ (of course this labeling need not be Sperner, but this will not be a problem as we will be applying the generalized form of the lemma). For a fine enough triangulation, P, of D, z^n winds the boundary of P around the origin n times. By the labeling, there are (0,0),(1,1),(2,2),(0,1),(1,2) and (2,0)-simplices which make up the boundary of P. Only the (0,1)-simplices contribute to $N(\partial P)$, and each time z^n winds the boundary around the origin, we find exactly one (0,1)-simplex, and so:

$$N(\partial P) = n$$

Per the usual progression, we form a sequence G_k ($k = 1, 2, \dots$) of triangulations which get finer. Here we require each triangle in G_k to have diameter $< \frac{1}{k}$. For each G_k, applying the generalized Sperner lemma (with $k = 2$), there exists some completely labeled triangle inside G. By the lemma, this means that for each vertex z_j of this triangle, $|p(z_j)| < \frac{2}{\sqrt{3}k}$.

Let y be an accumulation point of this sequence of triangles. We can make the value of $|p(z)|$ arbitrarily small by selecting z close enough to y, and so by continuity of polynomials, $p(y) = 0$.

□

The proofs of the Sperner Lemma which I have presented are not constructive, but there exist some proofs which give an explicit algorithm for locating the completely labeled triangles in a triangulated simplex (See Su, or Jarvis and Tanton). In light of this, the above proof of the fundamental theorem of algebra actually provides a computational framework for factoring a polynomial over the complex numbers (finding an approximate factorization at least). An algorithm would involve hierarchical subdivisions of a large disk in the complex plane, and would iteratively "zoom in" on a root with controllable error. [2]

5. Conclusion

The elegant proofs of the Brouwer Fixed-Point theorem and others show that the Sperner Lemma is not so innocent after all. It is a powerful tool that can be used in existence proofs. In constructive proofs of the lemma, Sperner even provides clean, elegant algorithms for solving the problem at hand. I mentioned the possibility of an algorithm for finding complex roots of a polynomial. In other works, the lemma has been used as a computational basis for many other problems, from finding fixed points of functions, to determining critial points in computational fluid dynamics.

Its strength perhaps lies in its ability to take facts which are true in discrete cases and take them to continuous scenarios. Consequently, it allows us to work with topological spaces in a purely combinatorial way.

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[2]A root-finding algorithm based on the Sperner lemma has never been implemented to my knowledge. Intuitively, it would have slower run time, since it must search for completely labeled simplices in dense triangulations, but in theory, its accuracy would not be very sensitive to the degree of the polynomial, whereas many root-finding algorithms lose their precision very fast as $\deg(p(z))$ gets high.